Aspects of Black Hole-Qubit Correspondence

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Contents

1 Introduction

Interestingly, the entanglement of qubits in quantum infromation and the entropy of supersymmetric black holes in string theory uses the same mathematics, Cayley's hyperdeterminant and Cartan's invariant. There are also surprising parallels in the way they can be classified: e.g. wrapping cycles corresponds to three-qubit basis vectors [1].

In this work, we will discuss Cayley's hyperdeterminant from a mathematical point of view, followed by some physical applications. We shall then express the entropy of a 24-charge $N = 4$ black hole (or the equivalent quantum entanglement problem) as a quartic polynomial of imaginary quaternions. We shall then discuss attempts to solve a more complicated 56-charge black hole problem, using imaginary octonions instead.

2 Cayley's Hyperdeterminant

2.1 Mathematical Background

The hyperdeterminant generalises the concept of the determinant to hypermatrices: multidimensional arrays of numbers (from any field) with dimension greater than two. It is a polynomial composed of the entries of the hypermatrix, which acts as a discriminant for the multilinear map represented by the hypermatrix (see [2] for full details). There is no known way to explicitly construct a hyperdeterminant given any hypermatrix of arbitrary dimensions. Fortunately, for our purposes we only need the hyperdeterminant of a $2 \times 2 \times 2$ hypermatrix, and for this we do have an explicit form, discovered by Arthur Cayley in 1845 [3].

Writing our $2 \times 2 \times 2$ hypermatrix as a_{ABC}, with $A,\ B,\ C$ taking values of either 0 or 1, Cayley's hyperdeterminant is defined as

$$Det\, a \equiv -\frac{1}{2}\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4} \tag{1}$$

where we have used the Einstein summation convention, $\varepsilon^{A_1A_2}$is the two-dimensional Levi-Civita symbol with $\varepsilon^{01} = 1$, and naturally $a_{A_iB_iC_i} \equiv a_{ABC}$ for $i = 1,\ 2,\ 3,\ 4$. These conventions will be followed throughout the paper. Explicitly it is

$$\begin{aligned} Det\, a \ \equiv\ & a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{011}^2a_{100}^2 \\ & -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\ & +a_{000}a_{011}a_{100}a_{111} + a_{001}a_{010}a_{101}a_{110} \\ & +a_{001}a_{011}a_{100}a_{110} + a_{001}a_{010}a_{100}a_{111} \\ & +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}). \end{aligned}$$

The required discriminant property is satisfied by the fact that the hyperdeterminant vanishes iff the following set of simultaneous equations in six unknowns x^A, y^B, z^Chas a non-zero solution:

$$\begin{aligned} a_{AB0}x^A y^B &= 0 \\ a_{AB1}x^A y^B &= 0 \\ a_{0BC}y^B z^C &= 0 \\ a_{1BC}y^B z^C &= 0 \\ a_{A0C}z^C x^A &= 0 \\ a_{A1C}z^C x^A &= 0. \end{aligned}$$

A property of the hyperdeterminant that will be useful later is that it is invariant under a cyclic exchange of A, B, C, also known as a triality. That is, if one performs this interchange on either $\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}$ or $a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4}$, $Det\, a$ remains unchanged. An elegant way to prove this is given in [1]: define three 2×2 matrices

$$\begin{aligned} \gamma^1(a)_{A_1A_2} &\equiv \varepsilon^{B_1B_2}\varepsilon^{C_1C_2}a_{A_1B_1C_1}a_{A_2B_2C_2} \\ \gamma^2(a)_{B_1B_2} &\equiv \varepsilon^{C_1C_2}\varepsilon^{A_1A_2}a_{A_1B_1C_1}a_{A_2B_2C_2} \\ \gamma^3(a)_{C_1C_2} &\equiv \varepsilon^{A_1A_2}\varepsilon^{B_1B_2}a_{A_1B_1C_1}a_{A_2B_2C_2}. \end{aligned}$$

Using the determinant formula

$$det\, A = \frac{1}{2}\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}a_{A_1A_2}a_{A_3A_4},$$

we have

$$det\, \gamma^1(a) = -Det\, a.$$

Now observe that $det\, \gamma^2(a)$ can be obtained from $det\, \gamma^1(a)$ by performing the following replacements on the $\varepsilon^{A_iA_j}$ terms:

$$\begin{aligned} A_i &\rightarrow B_i \\ B_i &\rightarrow C_i \\ C_i &\rightarrow A_i. \end{aligned}$$

Similarly, $det\, \gamma^3(a)$ is obtained by performing the remaining cyclic interchange. However, by manual calculation we can show

$$det\, \gamma^1(a) = det\, \gamma^2(a) = det\, \gamma^3(a)$$

and we have proved the triality. Note that this proof also demonstrates how the hyperdeterminant is in some sense the determinant of a determinant.

2.2 Physical Background

Here we shall briefly describe the physical applications of the hyperdeterminant to entanglement in quantum information and black hole entropy in the context of string theory. The results described are obtained from [1].

2.2.1 Qubit Entanglement

A qubit is a two-state quantum system, with basis state vectors $|0\rangle$ and $|1\rangle$. Any state vector in this system can be expressed as a linear combination of these basis vectors:|

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where α, $\beta \in \mathbb{C}$. If we impose the normalisation $|\alpha|^2 + |\beta|^2 = 1$, then we can interpret the coefficients as probability amplitudes of observing the corresponding basis states in the usual way.

It turns out that the hyperdeterminant is a convenient way to give the measure of tripartite entanglement of three qubits. Consider a three-qubit system whose state vectors are a linear composition of eight basis states:

$$|\psi\rangle = a_{ABC}\,|ABC\rangle$$

where $|000\rangle \equiv |0\rangle \otimes |0\rangle \otimes |0\rangle$ etc. and A, B, $C = 0, 1$ as usual. Then $Det\, a$ gives the tripartite entanglement of three qubits.[1] The problems detailed in 3.1 and 4.1 can be viewed as the tripartite entanglement of seven qubits.

2.2.2 Black Hole Entropy

In certain cases, supersymmetric black hole entropy as calculated in string theory is given by Cartan's invariant, of which Cayley's hyperdeterminant is a key component. The 24 real numbers of the problem in 3.1 can be viewed as 24 black hole charges, following which the invariant (2) becomes the entropy of a 24-charge black hole in $N = 4$ supersymmetry. Similarly, (14) is the entropy of a 56-charge black hole in $N = 8$ supersymmetry. [1]

3 Cartan over the Quaternions

3.1 Setup

We have 24 real numbers, labelled a_{ABD}, e_{EFA} and g_{GAC}, where A, B,...G are either 0 or 1. Let

$$I_1 \equiv a^4 + e^4 + g^4 + 2(a^2e^2 + e^2g^2 + g^2a^2) \tag{2}$$

with the individual terms defined as

$$\begin{aligned}
a^4 &\equiv \frac{1}{2}\varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\varepsilon^{D_1D_3}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4}\varepsilon^{D_2D_4}a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4} \qquad (3)\\
e^4 &\equiv \frac{1}{2}\varepsilon^{E_1E_2}\varepsilon^{F_1F_2}\varepsilon^{A_1A_3}\varepsilon^{E_3E_4}\varepsilon^{F_3F_4}\varepsilon^{A_2A_4}e_{E_1F_1A_1}e_{E_2F_2A_2}e_{E_3F_3A_3}e_{E_4F_4A_4}\\
g^4 &\equiv \frac{1}{2}\varepsilon^{G_1G_2}\varepsilon^{A_1A_2}\varepsilon^{C_1C_3}\varepsilon^{G_3G_4}\varepsilon^{A_3A_4}\varepsilon^{C_2C_4}g_{G_1A_1C_1}g_{G_2A_2C_2}g_{G_3A_3C_3}g_{G_4A_4C_4}\\
a^2e^2 &\equiv \frac{1}{2}\varepsilon^{A_1A_3}\varepsilon^{B_1B_2}\varepsilon^{D_1D_2}\varepsilon^{E_3E_4}\varepsilon^{F_3F_4}\varepsilon^{A_2A_4}a_{A_1B_1D_1}a_{A_2B_2D_2}e_{E_3F_3A_3}e_{E_4F_4A_4} \qquad (4)\\
e^2g^2 &\equiv \frac{1}{2}\varepsilon^{E_1E_2}\varepsilon^{F_1F_2}\varepsilon^{A_1A_3}\varepsilon^{G_3G_4}\varepsilon^{A_2A_4}\varepsilon^{C_3C_4}e_{E_1F_1A_1}e_{E_2F_2A_2}g_{G_3A_3C_3}g_{G_4A_4C_4}\\
g^2a^2 &\equiv \frac{1}{2}\varepsilon^{G_1G_2}\varepsilon^{A_1A_3}\varepsilon^{C_1C_2}\varepsilon^{A_2A_4}\varepsilon^{B_3B_4}\varepsilon^{D_3D_4}g_{G_1A_1C_1}g_{G_2A_2C_2}a_{A_3B_3D_3}a_{A_4B_4D_4}.
\end{aligned}$$

Note that $a^4$is the negative of Cayley's hyperdeterminant 1, treating a_{ABD} as a $2\times2\times2$ hypermatrix. Similar remarks obviously apply to $e^4$and g^4. As mentioned in 2.1, the triality of the hyperdeterminant means it does not matter which index is contracted with $\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}$ rather than $\varepsilon^{A_1A_2}\varepsilon^{A_3A_4}$. I_1 is Cartan's quartic invariant in $SL(2)\times SO(6,6)$. [1] The somewhat unusual labelling will be explained in 4.1.

Our objective is to express (2) as a linear combination of products of eight imaginary quaternions, taken four at a time, similar to (3) and (4). Since each imaginary quaternion can be viewed as a real vector space of dimension three, in some sense both sides have the same number of degrees of freedom ($8\times3=24$) and we can be optimistic about success.

We will also label these quaternions via the binary system (capital Roman letters), and write them as

$$x_{ABC} = x^i_{ABC}e_i \qquad (5)$$

where e_i are the standard imaginary quaternion basis, satisfying

$$e_ie_j = -\delta_{ij} + \varepsilon_{ijk}e_k \qquad (6)$$

with *i,j,k* taking values of 1,2,3.

3.2 Solution

It turns out one can directly assign the 24 a_{ABD} of (2) to the 24 quaternion components of (5) via

$$\begin{aligned}
x^1_{ABC} &\equiv a_{ABC}\\
x^2_{ABC} &\equiv e_{BCA} \qquad (7)\\
x^3_{ABC} &\equiv g_{CAB}
\end{aligned}$$

and

$$I_2 = \frac{1}{4}\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}x^i_{A_1B_1C_1}x^j_{A_2B_2C_2}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}(e_ie_je_ke_l + e_ie_ke_le_j). \tag{8}$$

Thus if we can prove that

$$I_1 = I_2 \tag{9}$$

then we have successfully defined this invariant over the imaginary quaternions.

3.3 Proof

Our proof will be slightly different from that found in [1]. Using the quaternion multiplication properties from (6) (note that quaternion multiplication is associative), we have

$$\begin{aligned} e_ie_je_ke_l &= (-\delta_{ij} + \varepsilon_{ijm}e_m)(-\delta_{kl} + \varepsilon_{kln}e_n) \\ &= \delta_{ij}\delta_{kl} - \delta_{ij}\varepsilon_{kln}e_n - \delta_{kl}\varepsilon_{ijm}e_m + \varepsilon_{ijm}\varepsilon_{kln}(-\delta_{mn} + \varepsilon_{mnp}e_p) \\ &= \delta_{ij}\delta_{kl} - \delta_{ij}\varepsilon_{kln}e_n - \delta_{kl}\varepsilon_{ijm}e_m - \varepsilon_{ijm}\varepsilon_{klm} + \varepsilon_{ijm}\varepsilon_{kln}\varepsilon_{pmn}e_p. \end{aligned}$$

Substituting the identity

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

yields

$$e_ie_je_ke_l = \delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \varepsilon_{ikl}e_j - \varepsilon_{jkl}e_i - \delta_{ij}\varepsilon_{klm}e_m - \delta_{kl}\varepsilon_{ijm}e_m.$$

Cycling the $j,\ k,\ l$ indices gives

$$e_ie_ke_le_j = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl} + \varepsilon_{ilj}e_k - \varepsilon_{klj}e_i - \delta_{ik}\varepsilon_{ljm}e_m - \delta_{jl}\varepsilon_{ikm}e_m$$

and we have

$$e_ie_je_ke_l + e_ie_ke_le_j = 2\delta_{ij}\delta_{kl} - 2\varepsilon_{jkl}e_i + \varepsilon_{ikl}e_j + \varepsilon_{ilj}e_k - (\delta_{ij}\varepsilon_{klm} + \delta_{kl}\varepsilon_{ijm} + \delta_{ik}\varepsilon_{ljm} + \delta_{jl}\varepsilon_{ikm})e_m. \tag{10}$$

Using only the first term, let

$$\begin{aligned} I_3 &= \frac{1}{4}\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}x^i_{A_1B_1C_1}x^j_{A_2B_2C_2}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}(2\delta_{ij}\delta_{kl}) \\ &= \frac{1}{2}\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}x^i_{A_1B_1C_1}x^i_{A_2B_2C_2}x^k_{A_3B_3C_3}x^k_{A_4B_4C_4} \\ &= \frac{1}{2}\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}(a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4} \\ &\quad + e_{B_1C_1A_1}e_{B_2C_2A_2}e_{B_3C_3A_3}e_{B_4C_4A_4} + g_{C_1A_1B_1}g_{C_2A_2B_2}g_{C_3A_3B_3}g_{C_4A_4B_4} \\ &\quad + 2(a_{A_1B_1C_1}a_{A_2B_2C_2}e_{B_3C_3A_3}e_{B_4C_4A_4} + e_{B_1C_1A_1}e_{B_2C_2A_2}g_{C_3A_3B_3}g_{C_4A_4B_4} \\ &\quad + g_{C_1A_1B_1}g_{C_2A_2B_2}a_{A_3B_3C_3}a_{A_4B_4C_4})) \end{aligned}$$

where we have used the definitions in (7) and the fact that $\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}$ is invariant under the interchange of $1 \leftrightarrow 2$, $3 \leftrightarrow 4$. By exploiting the triality of Cayley's hyperdeterminant from 2.1, we can see that the first three terms are a^2, $e^2$and $g^2$from (3), and the last three terms are ae, eg, and ga from (4). Thus we have

$$I_1 = I_3. \tag{11}$$

Hence proving (9) has been reduced to proving that the remaining eight terms in (10) vanish under contraction with $\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}x^i_{A_1B_1C_1}x^j_{A_2B_2C_2}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}$. We will do so by exploiting the symmetries in this polynomial. For ease of expression let

$$J^{ijkl} \equiv \varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}x^i_{A_1B_1C_1}x^j_{A_2B_2C_2}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}.$$

By inspection one can see that $\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}$ is invariant under the following interchanges:

$$\begin{aligned} 1 &\leftrightarrow 2,\ 3 \leftrightarrow 4 \\ 1 &\leftrightarrow 3,\ 2 \leftrightarrow 4 \\ 1 &\leftrightarrow 4,\ 2 \leftrightarrow 3. \end{aligned}$$

As the A_i are dummy indices, this implies J^{ijkl} is invariant under:

$$\begin{aligned} i &\leftrightarrow j,\ k \leftrightarrow l \\ i &\leftrightarrow k,\ j \leftrightarrow l \\ i &\leftrightarrow l,\ j \leftrightarrow k. \end{aligned}$$

Now we can show that the four terms of type $\varepsilon_{jkl}e_i$ cancel each other. Observe that $\varepsilon_{ikl}e_j - \varepsilon_{jkl}e_i$ is antisymmetric under the interchanges $i \leftrightarrow j$, $k \leftrightarrow l$, which implies that it will vanish under contraction with J^{ijkl}. This can be verified explicitly:

$$\begin{aligned} & \varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}x^i_{A_1B_1C_1}x^j_{A_2B_2C_2}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}(\varepsilon_{ikl}e_j - \varepsilon_{jkl}e_i) \\ = {} & \varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}(x^i_{A_1B_1C_1}x^j_{A_2B_2C_2} - x^j_{A_1B_1C_1}x^i_{A_2B_2C_2})\varepsilon_{ikl}e_j \\ = {} & \varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}(\varepsilon^{A_1A_3}\varepsilon^{A_2A_4} - \varepsilon^{A_1A_4}\varepsilon^{A_2A_3})x^i_{A_1B_1C_1}x^j_{A_2B_2C_2}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}\varepsilon_{ikl}e_j \\ = {} & \varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\varepsilon^{C_1C_2}\varepsilon^{C_3C_4}(\varepsilon^{A_1A_3}\varepsilon^{A_2A_4} - \varepsilon^{A_1A_3}\varepsilon^{A_2A_4})x^i_{A_1B_1C_1}x^j_{A_2B_2C_2}x^k_{A_3B_3C_3}x^l_{A_4B_4C_4}\varepsilon_{ikl}e_j \\ = {} & 0. \end{aligned}$$

Similarly, $\varepsilon_{ilj}e_k - \varepsilon_{jkl}e_i$ is antisymmetric under $i \leftrightarrow k$, $j \leftrightarrow l$, and its contribution also vanishes. Thus we have

$$J^{ijkl}(\varepsilon_{ikl}e_j + \varepsilon_{ilj}e_k - 2\varepsilon_{jkl}e_i) = 0. \tag{12}$$

Next we will consider the terms of type $\delta_{ij}\varepsilon_{klm}e_m$. Here each term will individually vanish: $\delta_{ij}\varepsilon_{klm}e_m$ and $\delta_{kl}\varepsilon_{ijm}e_m$ are both antisymmetric under $i \leftrightarrow j$, $k \leftrightarrow l$, and $\delta_{jl}\varepsilon_{ikm}e_m$ and $\delta_{ik}\varepsilon_{ljm}e_m$ are both antisymmetric under $i \leftrightarrow k$, $j \leftrightarrow l$. Thus we have

$$J^{ijkl}(\delta_{ij}\varepsilon_{klm} + \delta_{kl}\varepsilon_{ijm} + \delta_{ik}\varepsilon_{ljm} + \delta_{jl}\varepsilon_{ikm})e_m = 0. \tag{13}$$

Combining (11) with (12) and (13) gives us

$$I_1 = I_2$$

and we have proved the desired equality.

4 Cartan over the Octonions

4.1 Setup

Having successfully defined Cartan's invariant over eight imaginary quaternions, we will now tackle a more difficult problem. We will now construct a similar polynomial from 56 real numbers, and endeavour to express as a linear combination of products, taken four at a time, of eight imaginary octonions. An imaginary octonion can be viewed as a real vector space of dimension seven, so using the same degree of freedom argument from 3.1 ($8 \times 7 = 56$), we can again be optimistic about success.

Similar to 3.1, we will label the 56 real numbers as a_{ABD}, b_{BCE}, c_{CDF}, d_{DEG}, e_{EFA}, f_{FGB}, g_{GAC} with A, B, ...G either 0 or 1. Our polynomial has form

$$\begin{aligned} I_4 \equiv\ & a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 \\ & +2(a^2b^2 + a^2c^2 + a^2d^2 + a^2e^2 + a^2f^2 + a^2g^2 + b^2c^2 + b^2d^2 + b^2e^2 + b^2f^2 + b^2g^2 \\ & +c^2d^2 + c^2e^2 + c^2f^2 + c^2g^2 + d^2e^2 + d^2f^2 + d^2g^2 + e^2f^2 + e^2g^2 + f^2g^2 \\ & +8(abce + bcdf + cdeg + defa + efgb + fgac + gabd). \end{aligned} \tag{14}$$

Terms of the form a^4 are defined exactly as they were in (3), the negative of Cayley's hyperdeterminant. Terms of the form a^2b^2 are defined similar to (4), except that one must take care to contract the shared A, B, ...G index with the cross $\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}$ term, e.g.:

$$a^2b^2 \equiv \frac{1}{2}\varepsilon^{A_1A_2}\varepsilon^{B_1B_3}\varepsilon^{D_1D_2}\varepsilon^{B_2B_4}\varepsilon^{C_3C_4}\varepsilon^{E_3E_4}a_{A_1B_1D_1}a_{A_2B_2D_2}b_{B_3C_3E_3}b_{B_4C_4E_4}.$$

Note that each pair of a, b etc. has exactly one shared index, and all possible pairs are listed in (14). The new terms of form $abce$ are defined as

$$abce \equiv \varepsilon^{A_1A_4}\varepsilon^{B_1B_2}\varepsilon^{C_1C_2}\varepsilon^{D_1D_3}\varepsilon^{E_2E_4}\varepsilon^{F_3F_4}a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4} \tag{15}$$

i.e. one contracts over each shared index. Note that there are always six different pairs of shared indices in each *abce* term, and each of these seven terms excludes a different index (there are seven indices).

I_4 is Cartan's quartic invariant in $E_{7(7)}$, of which $SL(2) \times SO(6,6)$ is a maximal subgroup.[1] Note that we chose a_{ABD}, $e_{EFA}$and g_{GAC} to formulate the simpler problem in 3.1; this is related to using three imaginary octonions to construct the algebra of the imaginary quaternions.

As stated earlier, we now wish to define eight imaginary octonions in terms of the a_{ABD} etc. and express I_4 as a quartic polynomial of these octonions.

4.1.1 The Octonions

This will be a brief description of the octonions $\mathbb{O}$. They are an eight-dimensional normed division algebra over the real numbers, discovered by John T. Graves in 1843 and independently by Arthur Cayley in 1845 [4]. Writing their basis as e_0, e_1, ...e_7, $e_0 \equiv 1$ and the remaining octonions satisfy the following multiplication table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Table 1: Octonion multiplication table

where each entry is the product of the corresponding basis octonion in the first column with that in the first row, in that order. Note that

- $e_i^2 = -1$.
- $e_i e_j = -e_j e_i$ if $i \neq j$.
- If $e_i e_j = e_k$, $e_{i+1} e_{j+1} = e_{k+1}$ under addition modulo seven.

A more instructive way to write the multiplication table is the Fano plane, found in [4] and [1]. This demonstrates how there are seven ways to choose three imaginary octonions which from a subalgebra isomorphic to the imaginary quaternions. The octonions can also be constructed from the quaternions via the Cayley-Dickson construction [4].

Key properties of the octonions a, b, $c \in \mathbb{O}$ are as follows:

- normed: it has norm $|.| : \mathbb{O} \to \mathbb{R}$ in the vector space sense, satisfying $|ab| = |a|\,|b|$.
- division algebra: if $b \neq 0$ then there exists exactly one $x \in \mathbb{O}$ and exactly one $y \in \mathbb{O}$ such that $a = bx$ and $a = yb$.
- non-commutative: $ab \neq ba$, like the quaternions.
- non-associative: $(ab)c = a(bc)$.
- alternative: $a(ab) = (aa)b$ and $(ba)a = b(aa)$.

4.2 Unsatisfactory solution

One can construct 56 octonions, labelled x^i_{ABC} with $i = 1, 2, ...7$ and $A, B, C = 0, 1$ from our 56 real numbers a_{ABD} etc. via

$$\begin{aligned} x^1_{ABC} &\equiv a_{ABC}e_1 \\ x^2_{ABC} &\equiv b_{ABC}e_1 \end{aligned}$$

and so on. e_i are now basis octonions defined by the dual Fano plane, described in [1], and so the multiplication table will be different from that in 4.1.1.

Then define a polynomial very similar to (14):

$$I_5 \equiv \sum_{i=1}^{7}(x^i)^4 + \sum_{all\,pairs}(x^i)^2(x^j)^2 - 8(x^1x^2x^3x^5 + x^2x^3x^4x^6 + \ldots + x^7x^1x^2x^4)$$

where $(x^i)^4$, $(x^i)^2(x^j)^2$ and $x^ix^jx^kx^l$ are defined exactly the same way as in (3), (4) and (15). Note that double counting accounts for the missing factor of two on the $(x^i)^2(x^j)^2$ terms, and the new minus sign on the $x^ix^jx^kx^l$ terms.

The order of octonion multiplication makes no difference here. By inspection all the octonions will multiply out to 1 or -1, and by construction we have

$$I_4 = I_5.$$

This, however, is an inferior solution, since we have defined 56 imaginary octonions, rather than 8. Compare this to 3.3, where by exploiting symmetries we defined I_1 in terms of 8 imaginary quaternions rather than 24.

4.3 Outlook

This problem remains unsolved, largely due to the $x^ix^jx^kx^l$ terms. Note that due to the non-associativity of octonions, there are five ways to multiply a $e_ie_je_ke_l$ term.

References

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